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Random mixture of Ising systems of different spin values

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Abstract. The magnetic properties of the Ising mixture of a site model of A and B atoms, of which the magnetic moments, spins, concentrations and exchange energies are μ_A, μ_B, S_A, S_B, p_A, p_B, J_{AA}, J_{AB} and J_{BB}, are investigated. As examples, mixtures with $S=1$ and $S=\frac{1}{2}$, the former of which has the anisotropy D, are studied by the distribution function method in the Bethe approximation. The critical temperature (phase boundary between the paramagnetic (P)–ferromagnetic (F) and paramagnetic–antiferromagnetic (AF) phases), the energy and the zero-field susceptibility in the paramagnetic phase are obtained. They are expressed in terms of generalised Brillouin functions as a natural extension of the $S=\frac{1}{2}$ Ising model and the classical Heisenberg model.

1. Introduction

The magnetic properties of random systems have attracted considerable interest in recent years, and in particular the following works are mentioned in relation to the present work: Brout (1959)—distinction between quenched and annealed systems; Behringer (1957), Morgan and Rushbrooke (1961)—high-temperature and low-concentration expansions of dilute systems; Katsura and Tsujiyama (1966)—exact susceptibility and specific heat of the quenched and annealed one-dimensional dilute $S=\frac{1}{2}$ Ising systems; Matsubara *et al* (1973)—magnetisation processes of the one-dimensional dilute $S=\frac{1}{2}$ Ising ferromagnetic (F) and antiferromagnetic (AF) systems (see also Wortis (1974); Matsubara and Yoshimura (1973)—the one-dimensional dilute Ising system with higher spin; Matsubara (1974a), Katsura and Matsubara (1974)—F–AF binary mixture of the $S=\frac{1}{2}$ Ising site and bond systems, especially the paramagnetic (P) susceptibility and the phase boundary (see also Eggarter and Eggarter (1977), who also discussed the low-temperature mixed phase); Katsura (1975)—classical Heisenberg and planar models (see also Smith (1971), Tonegawa *et al* (1975) and Thorpe (1975) for the one-dimensional case only); Edwards and Anderson (1975), Matsubara and Sakata (1976)—existence of the spin glass phase (glass-like phase), the former in the infinitely long-range Gaussian bond mixture and the latter in the short-range binary bond mixture (a similar phase diagram to that of the latter was also obtained by Jayaprakash *et al* (1976)); Matsubara (1974b, c)—F–AF mixture by the distribution function method (see also Katsura (1977c), Katsura and Fujiki 1979); Katsura (1976, 1977b)—F–AF mixture by the low-field expansion and cumulant expansion methods; Sherrington and Kirkpatrick (1975), Katsura (1977a)—alternative derivation of the spin glass for the continuous distribution; Katsura and Fujiki (1979), thermodynamic properties of the binary mixture; Kudō *et al* (1978)—one-dimensional F–AF binary mixture of $S=1$ and $S=\frac{1}{2}$.†

In this paper we study a mixture of A and B atoms with spins $S_A=1$, $S_B=\frac{1}{2}$ and concentrations p_A, p_B in two and three dimensions as an example of a system with different spin values. The exchange energies are denoted by J_{AA}, J_{BB} and J_{AB}, each of which is either ferromagnetic or antiferromagnetic, and the anisotropy D is associated with A. The system is investigated by generalising the method used by Katsura and Fujiki (1979) and Kudō *et al* (1978, referred to hereafter as KMK). The distribution functions of the effective fields and effective anisotropies are introduced to reflect the fact that the random mixture has different expected spin values on each lattice site. The integral equation for the distribution function is derived, and the equations for the first and second moments of the effective fields are obtained. Using these relations, we obtain the energy and the susceptibility in the paramagnetic phase. The phase boundaries between P–F and P–AF phases are determined. The physical quantities are shown to be a generalisation of the $S=\frac{1}{2}$ Ising model and the classical Heisenberg model and are expressed in terms of (generalised) Brillouin functions. Numerical calculations of these boundaries together with the phase boundary between the spin glass phase and the paramagnetic phase will be given in a subsequent paper.

2. Bethe approximation for the random mixture with $S=1$ and $S=\frac{1}{2}$

We consider a cluster consisting of a central atom O under an external magnetic field H with its z nearest-neighbour atoms under an effective field H_i^*. An atom is either $\mathrm{A}(S=1)$ or $\mathrm{B}(S=\frac{1}{2})$. The Hamiltonian of the cluster is

$$\mathscr{H}=-2\sum_{i=1}^{z}J_{Oi}s_O s_i-\mu_O H s_O-\lambda_O s_O^2-\sum_{i=1}^{z}\mu_i H_i^* s_i-\sum_{i=1}^{z}\lambda_i^* s_i^2, \tag{2.1}$$

where $s=1, 0$ or -1 for the A atom and $\frac{1}{2}$ or $-\frac{1}{2}$ for the B atom, $J_{Oi}=J_{AA}$, J_{BB} or J_{AB}, $\mu=\mu_A$ or μ_B, $\lambda=\lambda_A$ or λ_B $(=0)$ represents the anisotropy. Introducing

$$\begin{array}{lll}
2\beta J_{AA}=K_{AA}, & \beta J_{AB}=K_{AB}, & \beta J_{BB}/2=K_{BB}, \\
\beta\mu_A H=C_A, & \beta\lambda_A=D_A, & \\
\beta\mu_B H/2=C_B, & \beta\lambda_B/4=D_B & \text{(for central spin)}, \\
\beta\mu_A H_i^*=L_{Ai}, & \beta\lambda_{Ai}^*=M_{Ai}, & \\
\beta\mu_B H_i^*/2=L_{Bi}, & \beta\lambda_{Bi}^*/4=M_{Bi} & \text{(for neighbouring spin)},
\end{array}$$

and

$$S=s \quad \text{for the A atom}, \qquad S=2s \quad \text{for the B atom},$$

we have

$$-\beta\mathscr{H}=\sum_{i=1}^{z}K_{Oi}S_O S_i+CS_O+DS_O^2+\sum_{i=1}^{z}L_i S_i+\sum_{i=1}^{z}M_i S_i^2, \tag{2.2}$$

† There is a problem as to whether the properties of spin systems of higher spins can be expressed in a unified way. The problem is partly answered in the molecular field and random phase approximations (Callen and Shitrikman 1965) by using the Brillouin function. In the exact solution and the Bethe approximation, however, such a property is not yet known. One of the motivations of the present study is to find a simple and general expression for the physical quantities of higher-spin systems in a unified way.

where $K_{Oi}=K_{AA}$, K_{BB} or K_{AB}, $C=C_A$ or C_B, $D=D_A$ or D_B $(=0)$, $L_i=L_{Ai}$ or L_{Bi}, $M_i=M_{Ai}$ or M_{Bi} $(=0)$ for a given configuration of the cluster. The effective field L_i and the effective anisotropy M_i will be determined self-consistently so that they take into account the effect of the outer spins. The density matrix of the cluster for a given configuration of neighbouring spins A and B (the numbers of A and B are k and $z-k$ respectively) is given by

$$\rho=\exp(CS_O+DS_O^2)\prod_{i=1}^{z}\rho_i, \qquad \rho_i=\exp[(K_{Oi}S_O+L_i)S_i+M_iS_i^2]. \tag{2.3}$$

The partial trace of ρ_i gives

$$\mathrm{tr}_i\rho_i=1+2u_i^{-1}\cosh(K_{Oi}S_O+L_i) \tag{2.4}$$

for the case where the ith spin is A, and

$$\mathrm{tr}_i\rho_i=2\cosh(K_{Oi}S_O+L_i) \tag{2.5}$$

for the case where the ith spin is B, where $u_i=\exp(-M_i)$. Equations (2.3) and (2.4) can be rewritten

$$\begin{aligned}(2.3)=\text{constant}\times\exp\bigg[&\frac{1}{2}\ln\left(\frac{u_i+2\cosh(K_{AA}+L_{Ai})}{\cosh(-K_{AA}+L_{Ai})}\right)S_O\\ &+\frac{1}{2}\ln\left(\frac{[u_i+2\cosh(K_{AA}+L_{Ai})][u_i+2\cosh(-K_{AA}+L_{Ai})]}{(u_i+2\cosh L_{Ai})^2}\right)S_O^2\bigg]\end{aligned} \tag{2.6}$$

for the central A spin,

$$\begin{aligned}(2.4)=\text{constant}\times\exp\bigg[&\frac{1}{2}\ln\left(\frac{\cosh(K_{AB}+L_{Bi})}{\cosh(-K_{AB}+L_{Bi})}\right)S_O\\ &+\frac{1}{2}\ln\left(\frac{\cosh(K_{AB}+L_{Bi})\cosh(-K_{AB}+L_{Bi})}{\cosh^2 L_{Bi}}\right)S_O^2\bigg]\end{aligned} \tag{2.7}$$

for the central A spin, and

$$(2.4)=\text{constant}\times\exp\left[\frac{1}{2}\ln\left(\frac{\cosh(K_{BB}+L_{Bi})}{\cosh(-K_{BB}+L_{Bi})}\right)S_O\right] \tag{2.8}$$

for the central B spin.

Thus the reduced density matrix of the cluster of the given configuration, $\mathrm{tr}''\rho$, reduced except at the central spin O and a neighbouring spin q, is given by

$$\mathrm{tr}''\rho=\text{constant}\times\exp(K_{Oq}S_OS_q+L_qS_q+M_qS_q^2+L'_qS_O+M'_qS_O^2), \tag{2.9}$$

where

$$L'_{Aq}=C_A+\frac{1}{2}\sum_{i=1,i\neq q}^{k'}\ln\left(\frac{u_i+2\cosh(K_{AA}+L_{Ai})}{u_i+2\cosh(-K_{AA}+L_{Ai})}\right)+\frac{1}{2}\sum_{j=k'+1,j\neq q}^{z}\ln\left(\frac{\cosh(K_{AB}+L_{Bj})}{\cosh(-K_{AB}+L_{Bj})}\right), \tag{2.10}$$

$$\begin{aligned}u'_q=\mathrm{e}^{-D}&\prod_{i=k'1,i\neq q}^{z}\frac{u_i+2\cosh L_{Ai}}{[u_i+2\cosh(K_{AA}+L_{Ai})][u_i+2\cosh(-K_{AA}+L_{Ai})]^{1/2}}\\ &\times\prod_{j=k'+1,j\neq q}^{z}\frac{\cosh 2L_{Bj}}{(\cosh(K_{AA}+L_{Bj})\cosh(-K_{AB}+L_{Bj}))^{1/2}}\end{aligned} \tag{2.11}$$

for the central spin A, and

$$L'_{Bq}=C_B+\sum_{i=1,i\neq q}^{k'}\ln\left(\frac{u_i+2\cosh(K_{BA}+L_{Ai})}{u_i+2\cosh(-K_{BA}+L_{Ai})}\right)+\sum_{j=k'+1,j\neq q}^{z}\ln\left(\frac{\cosh(K_{BB}+L_j)}{\cosh(-K_{BB}+L_j)}\right) \tag{2.12}$$

for the central spin B.

Here L'_q and M'_q are the effective field and effective anisotropy at the central spin resulting from the effects of the neighbouring spins 1, 2, ..., $q-1$, $q+1$, ..., z. $k'=k-1$ or k when the qth spin is A or B respectively. From (2.9) we see that if $L_q=L'_q$ and $M_q=M'_q$ then $\langle S_O\rangle=\langle S_q\rangle$. We require the averages (see Katsura and Fujiki 1979). The averages are carried out over the distribution functions $g_A(l_A, u)$ ($l_A\equiv\tanh L_A$, $u\equiv e^{-M}$) and $g_B(l_B)$ ($l_B=\tanh L_B$) and over all neighbouring configurations of A and B spins for given concentrations $p_A(=p)$ and $p_B=1-p$.

3. The integral equation for $g_A(l_A, u)$ and $g_B(l_B)$

First we consider the paramagnetic and the ferromagnetic phases. The probability that $\tanh L_A$ has a value between l_A and $l_A+\mathrm{d}l_A$ and e^{-M} has a value between u and $u+\mathrm{d}u$ is denoted by $g_A(l_A, u)\,\mathrm{d}l_A\,\mathrm{d}u$, and the probability that $\tanh L_B$ has a value between l_B and $l_B+\mathrm{d}l_B$ by $g_B(l_B)\,\mathrm{d}_B$. The distribution functions $g_A(l_A, u)$ and $g_B(l_B)$ are determined by the condition that the distribution of L_{Aq} and of L'_{Aq}, that of L_{Bq} and of $L_{B'q}$, and that of u_q and u'_q are the same, respectively. The integral equations for $g_A(l_A, u)$ and $g_B(l_B)$ read

$$\begin{aligned}
g_A(l_A, u)=&\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}\\
&\times\int\delta\biggl\{l_A-\tanh\biggl[C_A+\frac{1}{2}\sum_{i=1}^{k}\ln\left(\frac{u_i+2\cosh(K_{AA}+L_{Ai})}{u_i+2\cosh(-K_{AA}+L_{Ai})}\right)\\
&+\frac{1}{2}\sum_{j=k+1}^{z-1}\ln\left(\frac{\cosh(K_{AB}+L_{Bj})}{\cosh(-K_{AB}+L_{Bj})}\right)\biggr]\biggr\}\\
&\times\delta\biggl(u-e^{-D}\prod_{i=1}^{k}\frac{u_i+2\cosh L_{Ai}}{\{[u_i+2\cosh(K_{AA}+L_{Ai})][u_i+2\cosh(-K_{AA}+L_{Ai})]\}^{1/2}}\\
&\times\prod_{j=k+1}^{z-1}\frac{\cosh L_{Bj}}{[\cosh(K_{AB}+L_{Bj})\cosh(-K_{AB}+L_{Bj})]^{1/2}}\biggr)\\
&\times\prod_{i=1}^{k}g_A(l_{Ai}, u_i)\,\mathrm{d}l_{Ai}\,\mathrm{d}u_i\prod_{j=k+1}^{z-1}g_B(l_{Bj})\,\mathrm{d}l_{Bj}
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
g_B(l_B)=&\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}\\
&\times\int\delta\biggl\{l_B-\tanh\biggl[C_B+\frac{1}{2}\sum_{i=1}^{k}\ln\left(\frac{u_i+2\cosh(K_{BA}+L_{Ai})}{u_i+2\cosh(-K_{BA}+L_{Ai})}\right)
\end{aligned}$$

$$+\frac{1}{2}\sum_{j=k+1}^{z-1}\ln\left(\frac{\cosh(K_{BB}+L_{Bj})}{\cosh(-K_{BB}+L_{Bj})}\right)\Big]\Big\}$$

$$\times\prod_{i=1}^{k} g_A(l_{Ai}, u_i)\,dl_{Ai}\,du_i\prod_{j=k+1}^{z-1} g_B(l_{Bj})\,dl_{Bj} \tag{3.2}$$

(cf Matsubara 1974a, Katsura and Fujiki 1979, KMK 1978).

The coupled integral equations (3.1) and (3.2) for g_A and g_B can be solved by the iterative method and the solution becomes a multiple series.

In the case with no magnetic field, these exists a solution in which the distribution of l is the delta function at $l=0$. This is the paramagnetic solution, and we have

$$g_B(l)=\delta(l), \qquad g_A(l, u)=\delta(l)g_2(u), \tag{3.3}$$

$$g_2(u)=\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}$$

$$\times\int\delta\left(u-e^{-D}\prod_{i=1}^{k}\frac{u_i+2}{u_i+2\cosh K_{AA}}\frac{1}{(\cosh K_{AB})^{z-1-k}}\right)\prod_{i=1}^{k}g_2(u_i)\,du_i. \tag{3.4}$$

In particular in the one-dimensional case the solution reads

$$g_2(u)=(1-p)\sum_{i=0}^{\infty}p^i\delta(u-u^{(i)}), \tag{3.5}$$

where

$$u^{(0)}=1/\cosh K_{AB},$$
$$u^{(i+1)}=e^{-D}[(2+u^{(i)})/(2\cosh K_{AA}+u^{(i)})] \qquad (i=0, 1, 2, \ldots). \tag{3.6}$$

The integral equations (3.1) and (3.2) have different solutions from (3.3). One of the solutions describes the ferromagnetic state. The antiferromagnetic state is described by a solution of the generalisation of the integral equations (3.1) and (3.2) by taking the sublattice structure into consideration (cf Matsubara 1974a, Katsura 1977b, § 5).

Instead of l_A and u, the averages $x_1\equiv\langle S_{Ai}\rangle$ and $x_2\equiv\langle S_{Ai}^2\rangle$ can be used as independent variables of g_A. The integral equation of $g_A(x_1, x_2)$ for the one-dimensional system is given and discussed in KMK.

For the pure B case ($p=0$) the integral equation (3.2) has a solution

$$g_B(l)=\delta(l-l_B), \tag{3.7}$$

where $l_B(\equiv\tanh L_B)$ is determined by

$$L_B=C_B+(z-1)\tanh^{-1}(\tanh K_{BB}\tanh L_B). \tag{3.8}$$

For the pure A case ($p=1$) the integral equation (3.1) has a solution

$$g_A(l, u)=\delta(l-l_A)\delta(u=u_O), \tag{3.9}$$

where $l_A(\equiv\tanh L_A)$ and u_O are determined by

$$L_A=C_A+\frac{z-1}{2}\ln\left(\frac{u_O+2\cosh(K_{AA}+L_A)}{u_O+2\cosh(-K_{AA}+L_A)}\right),$$

$$u_O=\frac{e^{-D}(u_O+2\cosh L_A)^{z-1}}{\{[u_O+2\cosh(K_{AA}+L_A)][u_O+2\cosh(-K_{AA}+L_A)]\}^{(z-1)/2}}. \tag{3.10}$$

In the case of the antiferromagnetic phase, the lattice is divided into sublattices α and β, and we consider $g_A(l_A^\alpha)$, $g_B(l_B^\alpha)$, $g_A(l_A^\beta)$ and $g_B(l_B^\beta)$. In (3.1) and (3.2) l' on the right-hand side is replaced by $l_\alpha(l_\beta)$ and that on the left-hand side by $l_\beta(l_\alpha)$, and the equations form a closed set.

4. The energy

The energy per bond of the system, ϵ, is given by

$$-\beta\bar{\epsilon} = p^2 K_{AA}\overline{\langle S_A S'_A\rangle} + p(1-p)K_{AB}\overline{\langle S_A S'_B\rangle} + (1-p)^2 K_{BB}\overline{\langle S_B S'_B\rangle} + p\langle C_A S_A\rangle + (1-p)\langle C_B S_B\rangle, \tag{4.1}$$

where

$$\langle S_A S'_A\rangle = \int \left(\frac{2[e^K \cosh(L+L') - e^{-K}\cosh(L-L')]}{2[e^K\cosh(L+L')+e^{-K}\cosh(L-L')]+2u\cosh L+2u'\cosh L'+uu'}\right)_{K=K_{AA}} \times g_A(l,u)g_A(l',u')\,dl\,du\,dl'\,du', \tag{4.2}$$

$$\langle S_A S'_B\rangle = \int \left(\frac{e^K\cosh(L+L')-e^{-K}\cosh(L-L')}{e^K\cosh(L+L')+e^{-K}\cosh(L-L')+u\cosh L'}\right)_{K=K_{AB}} \times g_A(l,u)g_B(l')\,dl\,du\,dl', \tag{4.3}$$

$$\langle S_B S'_B\rangle = \int \left(\frac{e^K\cosh(L+L')-e^{-K}\cosh(L-L')}{e^K\cosh(L+L')+e^{-K}\cosh(L-L')}\right)_{K=K_{BB}} \tag{4.4}$$

and $\langle CS\rangle$ is given in (5.5).

In the paramagnetic region in zero field these expressions become

$$\langle S_A S'_A\rangle = \int B_1(K_{AA}, u+u'+uu'/2)g_2(u)g_2(u')\,du\,du', \tag{4.5}$$

$$\langle S_A S'_B\rangle = \int B_1(K_{AB}, u)g_2(u)\,du \tag{4.6}$$

and

$$\langle S_B S'_B\rangle = B_{1/2}(K_{BB}), \tag{4.7}$$

where B is the (generalised) Brillouin function defined by

$$B_1(x,u) = 2\sinh x/(u+2\cosh x), \qquad B_{1/2}(x) = \tanh x. \tag{4.8}$$

Special cases of $p_B = 1$, $p_A = 1$ and $H = 0$ in the one-dimensional case agree with the known results for the pure case of $S = \frac{1}{2}$ (Kramers and Wannier 1941), $S = 1$ (Katsura and Tsujiyama 1966, Suzuki *et al* 1967) and KMK (1978) respectively. The energy in the paramagnetic phase is independent of z in the Bethe approximation.

5. The susceptibility

First we consider the uniform susceptibility. The partition function of a cluster for a given configuration, which consists of a central spin A with k nearest neighbours A and

$z-k$ nearest neighbours B, can be shown to be

$$Z_{\rm A}=\exp(C_{\rm A}+D)\prod_{i=1}^{k}[1+2u_i^{-1}\cosh(K_{\rm AA}+L_{{\rm A}i})]\prod_{j=k+1}^{z}2\cosh(K_{\rm AB}+L_{{\rm B}j})$$
$$+\prod_{i=1}^{k}[1+2u_i^{-1}\cosh L_{{\rm A}i}]\prod_{j=k+1}^{z}2\cosh L_{{\rm B}j}$$
$$+\exp(-C_{\rm A}+D)\prod_{i=1}^{k}[1+2u_i^{-1}\cosh(-K_{\rm AA}+L_{{\rm A}i})]$$
$$\times\prod_{j=k+1}^{z}2\cosh(-K_{\rm AB}+L_{{\rm B}j}), \tag{5.1}$$

and that for a central spin B is

$$Z_{\rm B}={\rm e}^{C_{\rm B}}\prod_{i=1}^{k}[1+2u_i^{-1}\cosh(K_{\rm BA}+L_{{\rm A}i})]\prod_{j=k+1}^{z}2\cosh(K_{\rm BB}+L_{{\rm B}j})$$
$$+{\rm e}^{-C_{\rm B}}\prod_{i=1}^{k}[1+2u_i^{-1}\cosh(-K_{\rm BA}+L_{{\rm A}i})]\prod_{j=k+1}^{z}2\cosh(-K_{\rm BB}+L_{{\rm B}j}). \tag{5.2}$$

The averages of the central spins in these configurations are given by

$$\langle S_{\rm A}\rangle=B_1\bigg[C_{\rm A}+\sum_{i=1}^{k}\frac{1}{2}\ln\bigg(\frac{u_i+2\cosh(K_{\rm AA}+L_{{\rm A}i})}{u_i+2\cosh(-K_{\rm AA}+L_{{\rm A}i})}\bigg)+\prod_{j=k+1}^{z}\frac{1}{2}\ln\bigg(\frac{\cosh(K_{\rm AB}+L_{{\rm B}j})}{\cosh(-K_{\rm AB}+L_{{\rm B}j})}\bigg),$$
$$\mathrm{e}^{-D}\prod_{i=1}^{k}\frac{u_i+2\cosh L}{\{[u_i+2\cosh(K_{\rm AA}+L_{{\rm A}i})][u_i+2\cosh(-K_{\rm AA}+L_{{\rm A}i})]\}^{1/2}}$$
$$\times\prod_{j=k+1}^{z}\frac{\cosh L_{{\rm B}j}}{[\cosh(K_{\rm AB}+L_{{\rm B}j})\cosh(-K_{\rm AB}+{\rm L}_{{\rm B}j})]^{1/2}}\bigg] \tag{5.3}$$

and

$$\langle S_{\rm B}\rangle={\rm B}_{1/2}\bigg[C_{\rm B}+\sum_{i=1}^{k}\frac{1}{2}\ln\bigg(\frac{u_i+2\cosh(K_{\rm BA}+L_{{\rm A}i})}{u_i+2\cosh(-K_{\rm BA}+L_{{\rm A}i})}\bigg)+\sum_{j=k+1}^{z}\frac{1}{2}\ln\bigg(\frac{\cosh(K_{\rm BB}+L_{{\rm B}j})}{\cosh(-K_{\rm BB}+L_{{\rm B}j})}\bigg)\bigg]. \tag{5.4}$$

The magnetisation of the central spin averaged over all configurations is

$$\langle\mu S_{\rm O}\rangle=\sum_{k=0}^{z}\binom{z}{k}p^k(1-p)^{z-k}\int[\mu_{\rm A}p\langle S_{\rm A}\rangle+\tfrac{1}{2}\mu_{\rm B}(1-p)\langle S_{\rm B}\rangle]$$
$$\times\prod_{i=1}^{k}g_{\rm A}(l_{{\rm A}i},u_i)\,{\rm d}l_{{\rm A}i}\,{\rm d}u_i\prod_{j=k+1}^{z}g_{\rm B}(l_{{\rm B}j})\,{\rm d}l_{{\rm B}j}. \tag{5.5}$$

In the low-field limit in the paramagnetic phase, where $L_{\rm A}\to 0$ and $L_{\rm B}\to 0$ ($M\neq 0$), the first argument of B_1 in (5.3) is

$$C_{\rm A}+\sum_{i=1}^{k}\frac{2\sinh K_{\rm AA}}{u_i+2\cosh K_{\rm AA}}L_{{\rm A}i}+\sum_{j=k+1}^{z}(\tanh K_{\rm AB})L_{{\rm B}j} \tag{5.6}$$

and the second argument of B_1 in (5.3) is

$$\mathrm{e}^{-D}\prod_{i=1}^{k}\frac{u_i+2}{u_i+2\cosh K_{\rm AA}}(\cosh K_{\rm AB})^{z-k}. \tag{5.7}$$

Noting that

$$\frac{1}{2}\ln\left(\frac{u+2\cosh(K+L)}{u+2\cosh(-K+L)}\right)\to B_1(K,u)L \tag{5.8}$$

for $L\to 0$, we have from (2.10), (2.11) and (2.12)

$$L'_{Aq}=C_A+\sum_{i=1,i\neq q}^{k} B_1(K_{AA},u_i)L_{Ai}+\sum_{j=k'+1}^{z} B_{1/2}(K_{AB})L_{Bj}, \tag{5.9}$$

$$L'_{Bq}=C_B+\sum_{i=1}^{k'} B_1(K_{BA},u_i)L_{Ai}+\sum_{j=k'+1,j\neq q}^{z} B_{1/2}(K_{BB})L_{Bj}, \tag{5.10}$$

$$u'_q=\mathrm{e}^{-D}\prod_{\substack{i=1\\ i\neq q}}^{k'}\frac{u_i+2}{u_i+2\cosh K_{AA}}\frac{1}{(\cosh K_{AB})^{z-k'}}. \tag{5.11}$$

Since $B_1(x,y)\simeq 2x/(y+2)$ for $x\to 0$, we have from (5.3) and (5.4)

$$\langle S_A\rangle=\left(1+\frac{\mathrm{e}^{-D}}{2}\prod_{i=1}^{k}\frac{u_i+2}{u_i+2\cosh K_{AA}}\frac{1}{(\cosh K_{AB})^{z-k}}\right)^{-1}$$
$$\times\left(C_A+\sum_{i=1}^{k} B_1(K_{AA},u_i)L_{Ai}+\sum_{j=k+1}^{z} B_{1/2}(K_{AB})L_{Bj}\right), \tag{5.12}$$

$$\langle S_B\rangle=C_B+\sum_{i=1}^{k} B_1(K_{BA},u_i)L_{Ai}+\sum_{j=k+1}^{z} B_{1/2}(K_{BB})L_{Bj}. \tag{5.13}$$

Now we carry out the average over all possible configurations of the clusters. The distribution functions g_A and g_B are regarded as independent of the site. Then the summations in (5.9)–(5.13) are replaced by the average value multiplied by the number of terms (k' or $z-k'$). For example

$$\overline{\textstyle\sum_{i=1,i\neq q}^{k+1} B_1(K_{AA},u_i)L_{Ai}}$$
$$=\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}\int\sum_{i=1,i\neq q}^{k+1} B_1(K_{AA},u_i)L_{Ai}$$
$$\times\prod_{\substack{i=1\\ i\neq q}}^{k+1} g_A(l_{Ai},u_i)\,\mathrm{d}l_{Ai}\,\mathrm{d}u_i$$
$$=\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}\sum_{\substack{i=1\\ i\neq q}}^{k+1}\int B_1(K_{AA},u_i)L_{Ai}g_A(l_{Ai},u_i)\,\mathrm{d}l_{Ai}\,\mathrm{d}u_i$$
$$=\overline{B_1(K_{AA},u)}\,\overline{L_A}\sum_{k=0}^{z-1}\binom{z-1}{k}p^k(1-p)^{z-1-k}k$$
$$=(z-1)pB_1(K_{AA},\bar{u})\overline{L_A}. \tag{5.14}$$

Here we have approximated $\overline{B_1(K,u)}=B_1(k,\bar{u})$.

Then the requirements $\overline{L_q}=\overline{L'_q}$ and $\bar{u}_q=\overline{u'_q}$ are transformed into

$$\overline{L_A}=C_A+(z-1)pB_1(K_{AA},\bar{u})\overline{L_A}+(z-1)(1-p)B_{1/2}(K_{AB})\overline{L_B}, \tag{5.15}$$

$$\overline{L_B}=C_B+(z-1)pB_1(K_{BA},\bar{u})\overline{L_A}+(z-1)(1-p)B_{1/2}(K_{BB})\overline{L_B},$$

$$\bar{u}=e^{-D}\left(\frac{\bar{u}+2}{\bar{u}+2\cosh K_{AA}}\right)^{p(z-1)}\left(\frac{1}{\cosh K_{AB}}\right)^{(1-p)(z-1)} \tag{5.16}$$

The zero-field susceptibility is given from (5.12)–(5.16) by

$$\frac{kT\chi}{N}=(p_A\mu'_A\ \tfrac{1}{2}p_B\mu_B)\left[\begin{pmatrix}\partial C_A/\partial\beta\\ \partial C_B/\partial\beta\end{pmatrix}+z\begin{pmatrix}B_1(K_{AA},\bar{u}) & B_{1/2}(K_{AB})\\ B_1(K_{BA},\bar{u}) & B_{1/2}(K_{BB})\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}p_A & 0\\ 0 & p_B\end{pmatrix}\begin{pmatrix}\partial L_A/\partial\beta H\\ \partial L_B/\partial\beta H\end{pmatrix}\right], \tag{5.17}$$

where

$$\mu'_A=\mu_A\left[1+\frac{e^{-D}}{2}\left(\frac{\bar{u}+2}{\bar{u}+2\cosh K_{AA}}\right)^{zp}\left(\frac{1}{\cosh K_{AB}}\right)^{z(1-p)}\right]^{-1}$$

$$=\mu_A(1+\tfrac{1}{2}e^{-D}\bar{u}^{z/(z-1)})^{-1}. \tag{5.18}$$

Here we used

$$\sum_{k=0}^{z}\binom{z}{k}kp^k(1-p)^{z-k}=zp.$$

$\partial L_A/\partial\beta H$ and $\partial L_B/\partial\beta H$ in the low-field limit are obtained from (5.8)–(5.10):

$$\begin{pmatrix}\partial\overline{L_A}/\partial\beta H\\ \partial\overline{L_B}/\partial\beta H\end{pmatrix}=\begin{pmatrix}\partial C_A/\partial\beta H\\ \partial C_B/\partial\beta H\end{pmatrix}+(z-1)\begin{pmatrix}B_1(K_{AA},\bar{u}) & B_{1/2}(K_{AB})\\ B_1(K_{AB},\bar{u}) & B_{1/2}(K_{BB})\end{pmatrix}$$

$$\times\begin{pmatrix}p_A & 0\\ 0 & p_B\end{pmatrix}\begin{pmatrix}\partial\overline{L_A}/\partial\beta H\\ \partial\overline{L_B}/\partial\beta H\end{pmatrix}.$$

Hence

$$\begin{pmatrix}\partial\overline{L_A}/\partial\beta H\\ \partial\overline{L_B}/\partial\beta H\end{pmatrix}=\begin{pmatrix}1-(z-1)B_1(K_{AA},\bar{u})p_A & -(z-1)B_{1/2}(K_{AB})p_B\\ -(z-1)B_1(K_{BA},\bar{u})p_A & 1-(z-1)B_{1/2}(K_{BB})p_B\end{pmatrix}^{-1}\begin{pmatrix}\mu_A\\ \frac{1}{2}\mu_B\end{pmatrix}. \tag{5.20}$$

The susceptibility in the paramagnetic phase is given by (5.17) with (5.20), i.e.

$$\frac{kT\chi}{N}=(\mu'_A p_A\tfrac{1}{2}\mu_B p_B)\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+z\begin{pmatrix}B_1(K_{AA},\bar{u})p_A & B_{1/2}(K_{AB})p_B\\ B_1(K_{BA},\bar{u})p_B & B_{1/2}(K_{BB})p_B\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}1-(z-1)B_1(K_{AA},\bar{u})p_A & -(z-1)B_{1/2}(K_{AB})p_B\\ -(z-1)B_1(K_{BA},\bar{u})p_A & 1-(z-1)B_{1/2}(K_{BB})p_B\end{pmatrix}^{-1}\right]\begin{pmatrix}\mu_A\\ \frac{1}{2}\mu_B\end{pmatrix}. \tag{5.21}$$

In particular, the critical temperature is given by

$$[1-(z-1)B_1(K_{AA},\bar{u})p_A][1-(z-1)B_{1/2}(K_{BB})p_B]$$

$$-(z-1)^2B_1(K_{BA},\bar{u})B_{1/2}(K_{AB})p_Ap_B=0, \tag{5.22}$$

where $\bar{u}$ is determined from (5.16). We have assumed that the transition is second order.

In the case in which we consider the sublattice structure of the lattice, the left-hand sides of equations (5.1)–(5.7), (5.9), (5.10), (5.12), (5.13), (5.15), (5.19) and (5.20) are regarded as the ones with $\alpha(\beta)$, and the right-hand sides of these equations with $\beta(\alpha)$.

Two sets of equations (5.15) are decoupled by a similarity transformation as in Katsura *et al* (1979), and the uniform and staggered susceptibility obtained. The former is the same as in equations (5.21) and (5.22), and the latter has the same form but the signs of K_{AA}, K_{BB} and K_{AB} are reversed. Equations (5.21) and (5.22) are generalisations of the corresponding results for the Ising mixture of $S=\frac{1}{2}$ and for the classical Heisenberg mixture (Katsura and Matsubara 1974, Katsura 1975). The susceptibility in the pure limit $p_A = 1$ in the one-dimensional case,

$$\frac{kT\chi}{N}=\frac{2(u+2\cosh K_{AA})^2}{e^{-D}(u+2)^2+2(u+2\cosh K_{AA})^2}\,\frac{1+B_1(K_{AA},u)}{1-B_1(K_{AA},u)},$$
$$u=\tfrac{1}{2}\{e^{-D}-2\cosh K_{AA}+[(2\cosh K_{AA}-e^{-D})^2+8e^{-D}]^{1/2}\},\tag{5.23}$$

agrees with known results (Katsura and Tsujiyama 1966, Suzuki *et al* 1967). The critical temperature for $p_A = 1$ for arbitrary z is given by $B_1(K,u) = 1/(z-1)$, i.e.

$$e^D\frac{2[2(z-1)\tanh(K/2)-1-\tanh^2(K/2)]}{1-\tanh^2(K/2)}$$
$$=\left(1-\frac{\tanh(K/2)}{z-1}\right)^{z-1}\qquad(K=K_{AA}).\tag{5.24}$$

The case $D=0$ agrees with known results (Obokata and Oguchi 1968).

When we approximate $B_{1/2}(K)\sim K$ and $B_1(K,u)\sim\frac{2}{3}K$, and replace $z-1$ by z in (5.22), we obtain the critical temperature in the molecular field approximation:

$$2kT_c/z=\tfrac{1}{2}\{\tfrac{1}{3}J_{AA}p_A+J_{BB}p_B\pm[(\tfrac{2}{3}J_{AA}p_A-J_{BB}p_B)^2+\tfrac{8}{3}J_{AB}^2p_Ap_B]^{1/2}\}.\tag{5.25}$$

By considering the staggered susceptibility, the Néel temperature is obtained from equations (5.22) and (5.25) after replacing K by $-K$.

In the one-dimensional antiferromagnetic mixture $J_{AA}<0$, $J_{BB}<0$, $J_{AB}\gtrless 0\ (\neq 0)$, the susceptibility diverges for $T\to 0$, except at $p_A = 1$ or $p_B = 1$ (Matsubara 1974b) or $\mu_A = \mu_B/2$.

$$\chi/N\simeq(\mu_A^2-\mu_B^2/4)p(1-p)/kT.\tag{5.26}$$

6. Conclusions

The magnetic properties of a random Ising model with $S=1$ and $S=\frac{1}{2}$, in which the former has anisotropy D, were investigated using the Bethe approximation and distribution function methods. The energy, the uniform and staggered susceptibilities in the paramagnetic phase, and the phase boundaries between P–F and P–AF phases were obtained.

The result holds exactly in the one-dimensional system and reproduces the results of KMK (1979). Our method gives a generalisation of the $S=\frac{1}{2}$ Ising model and the classical Heisenberg model. The properties of a random mixture of Ising systems with general higher spins are expected to be expressed in terms of generalised Brillouin functions in a similar way. The discussion on the spin glass and mixed phases will be published in the near future together with numerical calculations for the phase boundaries between P–F, P–AF and P–spin glass phases.

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